Break Divisors as Canonical Representatives for Divisor Classes on Complete Graphs: Applications to the Internet of Things

Abstract

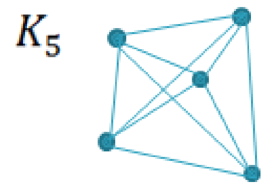
Network flow optimality on directed graphs is a central topic in graph theory and combinatorics. Given a complete graph G, we investigate the relationship between the score vectors of special tournaments on G and its spanning trees. Score vectors are assigned to tournaments on directed graphs in which the champion is a vertex A such that there is a directed path from A to every other vertex B. Comparing patterns examined between generated score vector sequences and parking functions, we define *break divisors* on complete graphs with n vertices to be n-tuples characterized by two distinct properties – one that defines the number of potential score vectors, and the other that accounts for impossible score sequences. We also show that these break divisors function as canonical representatives for each of the n^{n-2} linear equivalence classes of divisors of degree g on G. Specializing for complete graphs, we present an efficient algorithm for computing break divisors directly from spanning trees through the construction of partial graph orientations. This geometric bijection completes an alternate, unique proof of Arthur Cayley's famous tree formula. The research results, particularly the concrete characterization of break divisors as mathematical tools to count and identify spanning trees and score vectors of tournaments with transitive champions, can have a multitude of applications to the development of Internet of Things frameworks.

Agni Kumar

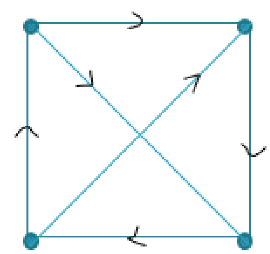
Introduction

This research work delves into a topic within the mathematical field of graph theory and combinatorics. Below we provide definitions of several general terms that are relevant to the content of the report. The definitions of more specific and specialized terms, characterizing the nature of this research, are provided in the "Preliminaries" section.

Complete Graphs. In the mathematical field of graph theory, a *complete graph* is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. We denote complete graphs on n vertices as K_n. Complete graphs have $\binom{n}{2}$ edges. In this paper, we study the mathematics of *break divisors* on complete graphs, taking advantage of their symmetries – one example being to treat all edges as equivalent.



Digraphs. A directed graph (*digraph*) is a graph whose edges have a direction associated with them. A directed graph having no multiple edges or loops is called a simple directed graph. Here we study *oriented graphs*, which are digraphs containing no bidirected edges. Specifically, we look into *complete oriented graphs*, in which each pair of vertices is joined by a single edge having a unique direction. We define such graphs to be *tournaments*.


Digraph of K_4

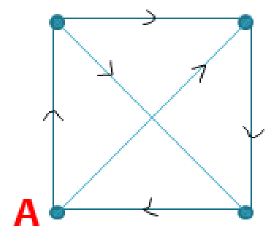
Preliminaries

In this section, we introduce definitions and concepts referenced throughout the paper and their connections to Arthur Cayley's famous tree formula, for which we will provide a new proof. In a later section, we also show that there are exactly n^{n-2} score vectors of tournaments in which a given player is a transitive champion.

Tournaments. A tournament is a directed graph obtained by assigning a direction for each edge. For the purposes of this research, *tournaments* are considered to be orientations of complete graphs. Here, a tournament is structured to interpret the outcome of a round-robin tournament: every player encounters every other player exactly once, and no draws occur.

Tournament Digraphs. Modeling a tournament as a directed graph, vertices correspond to players, and edges between each pair of players is oriented from the winner to the loser. We say that if player A beats player B, then A *dominates* B. As there are two possible orientations for each edge, there are $2^{\binom{n}{2}}$ total possible tournaments on K_n.

Transitive Champions. Regarding a tournament as a directed version of K_n, we define the *transitive* property to mean that whenever there is an arrow from a vertex A to a vertex B and an arrow from B to a vertex C, there is also an arrow from A to C. In this paper, a *transitive champion* is a vertex A such that there is a directed path from A to every other vertex B. Tournaments in which a vertex q is a transitive champion are termed *q-connected orientations*.[1]



transitive champion A

Score Vectors. For a given vertex, we assign the number of head arrows adjacent (i.e. pointing) to it as the *indegree* of the vertex and the number of tail arrows adjacent to it as its *outdegree*. For this paper, we have chosen to define "score vectors" as the outdegree counts (wins) of each vertex in clockwise order starting from vertex A (see **Fig. 1** below).

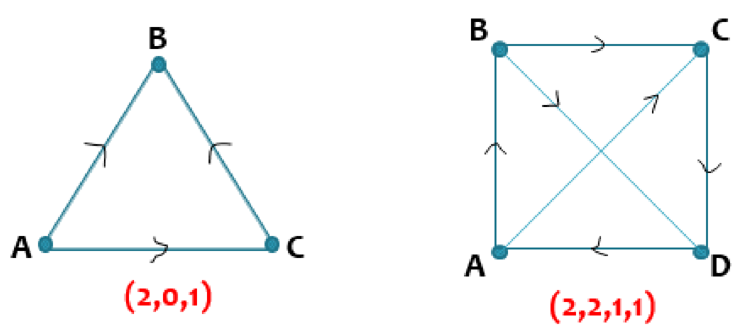


Figure 1. Here we show two tournaments with vertex A as the transitive champion and their corresponding score vector notations detailing each vertex's total number of wins.

Spanning Trees. A spanning tree T of an undirected graph G is a connected subgraph that includes all of the vertices of G in which there are no cycles. In general, a graph may have several spanning trees, though an unconnected graph will not contain a spanning tree. (The properties of spanning trees, specifically on complete graphs, are discussed thoroughly in the "Mathematical Tools and Methods" section – Cayley's Formula.)

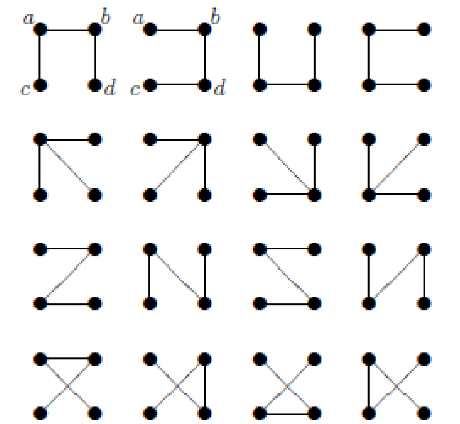


Figure 2. 16 total spanning trees on K_4.

Parking Functions. For the purposes of alignment with the upcoming definition of break divisors, we define a parking function of length $n-1$ to be a sequence $(a_0, a_1, \dots, a_{n-2})$ of nonnegative integers such that its non-decreasing rearrangement $(b_0, b_1, \dots, b_{n-2})$ satisfies $b_i \leq i$ for all i. We denote by ρ_n the set of all parking functions of length n.[3]

Chip-firing. There are many variations to the chip-firing game. Typically, we let $G = (V, E)$ be a finite connected undirected graph without loops. The *chip configuration* is denoted by the assignment of a nonnegative integer $c(v)$ number of chips to each vertex v of G.[4] Letting N denote the total number of chips, we have $\sum_{v \in V(G)} c(v) = N$.

If a vertex v has at least as many chips on it as its degree, i.e. $c(v) \geq deg(v)$, we say that v is *ready to fire*. A vertex v that is ready to fire is able to send chips to its neighbors by sending one chip along each of its incident edges. In turn, the value of each neighbor is increased by one and the value at the vertex that fired is decreased by its degree. A chip configuration is *stable* if no vertex is ready to fire, that is, if $c(v) < deg(v)$ for all vertices $v \in V(G)$. Two chip configurations are considered *equivalent* if one chip configuration can transform into the other through a series of firings and reverse-firings.

Research Objective

The objective of this research work is to provide a new and simple proof of Arthur Cayley's famous tree formula through the use of the Cori-LeBorgne result (defined in the "Mathematical Tools and Methods" section), while also showing that there are exactly n^{n-2} score vectors of tournaments in which a given player is a transitive champion. We do this by finding a bijection between spanning trees and *break divisors* on complete graphs through the establishment of an explicit inverse map. An outline is given below:

1. Using the Cori-LeBorgne result, we show that there are exactly n^{n-2} equivalence classes of divisors of any given degree (e.g. degree 0).
2. Next, we show that there are exactly n^{n-2} different break divisors on K_n by showing that there is exactly one break divisor in each equivalence class.
3. Finally, we show that the correspondence between spanning trees and break divisors is really a bijection, by finding an explicit inverse map. We apply this research, and the interesting connections revealed through the process, to the theory of network flow on directed graphs.

Mathematical Tools and Methods

In this section, we introduce the mathematical tools necessary to fulfill the above outline.

Cayley's Formula (1889). The number of spanning trees in a complete graph K_n is n^{n-2}.

Break Divisors. To define the concept of *break divisors*, we set up a scenario – let T be a spanning tree of a graph G. Let $e_1, \ldots, e_g$ be the edges not in T. For each i, choose a vertex v_i as one of the two endpoints of e_i. Assign a single chip to each such v_i. Let $D = v_1 + \cdots + v_g$ be the corresponding divisor. This will be a break divisor, and every break divisor has this form.[9]

(We can get coefficients bigger than 1 in D because the same vertex v can be an endpoint of more than one edge e_i.) As we use them throughout this paper, break divisors can also be defined as sequences of length n that fit the following properties[5]:

(i) Entries sum to $\binom{n-1}{2}$.

(ii) When the entries of D are written in non-decreasing order, then for each $k = 1, \dots, n$, the partial sum $s_k = a_1 + \cdots + a_k$ should satisfy $s_k \geq \binom{k-1}{2}$.

(i) and (ii) above imply that $0 \leq a_i, b_i \leq n-2$: taking $k = 1$, $s_1 \geq \binom{0}{2} = 0$. Furthermore, we know that $s_{n-1} \geq \binom{n-1-1}{2} = \binom{n-2}{2}$ and $s_n = \binom{n-1}{2}$. Therefore, the maximum value of an a_i, b_i is $\binom{n-1}{2} - \binom{n-2}{2} = n-2$. (The concept of *break divisors* was first introduced in 2008 as an abstract way to characterize tropical curves[12]. However, the characterization of these break divisors was not concrete; they could not be described with a set of properties and thus could not be directly counted or easily applied to general situations. That break divisors, as defined previously, are in bijection with sequences satisfying (i) and (ii) can be verified by the combination of Theorem 4.8 and Proposition 4.11 in [ABKS], in addition to specialization to the case $G = K_n$.)

Cori-LeBorgne result (2014).[6] $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ on K_n are chip-firing equivalent if and only if the sum of a_i equals the sum of b_i and for all i, j we have that

$$a_i - a_j \equiv b_i - b_j \pmod{n}.$$

Mixed graphs (partial orientations). A mixed graph $G = (V, E, A)$ is a mathematical object consisting of a set of vertices V, a set of undirected edges E, and a set of directed edges A. (See the "Appendix" for applications of mixed graphs in establishing a geometric bijection between break divisors and spanning trees on complete graphs.)

Results and Discussion

As we are investigating several smaller problems to reach the overarching goal, we have broken up the research results into multiple parts. Below, we examine the total number of *unique* (i.e. rearrangements within sequences not included) score vectors of tournaments for which vertex A is a transitive champion out of the total possible number of tournament digraphs on K_n. What are considered to be identical score sequences, applicable on K_n for $n \geq 3$, serve to explain the fact that different tournaments can have the same score vector.

1 ***Score vectors for tournaments in which A is a transitive champion.*** Beginning with small values of n, we see the following emerging patterns:

- As shown in **Fig. 3** below, for $n = 2$, we see that there is $2^{2-2} = 1$ score vector (which happens to be "unique") in which A is a transitive champion among the total $2^{\binom{2}{2}} = 2$ tournaments.

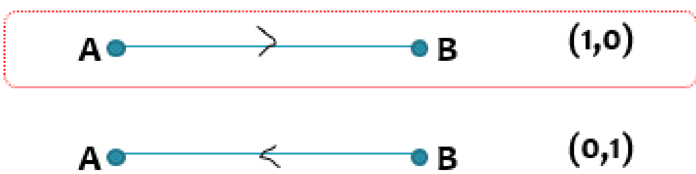


Figure 3. Tournaments on K_2; red box marks score vector.

- As shown in **Fig. 4**, for $n = 3$, we see that there are $3^{3-2} = 3$ unique score vectors in which A is a transitive champion among the total $2^{\binom{3}{2}} = 8$ tournaments.

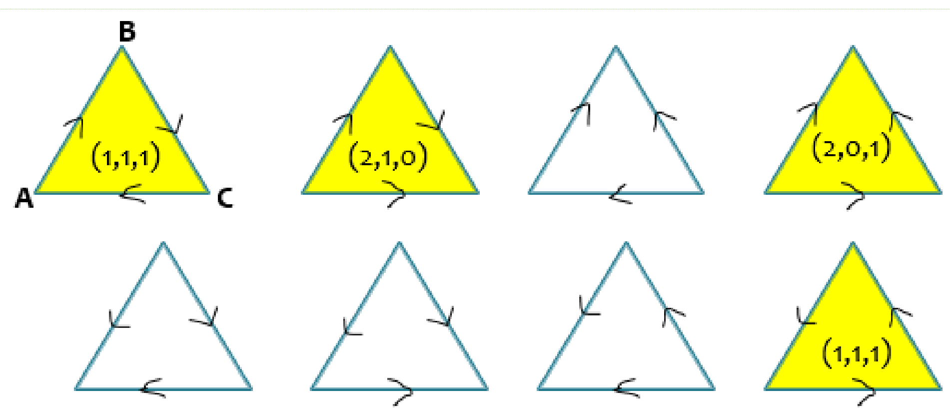
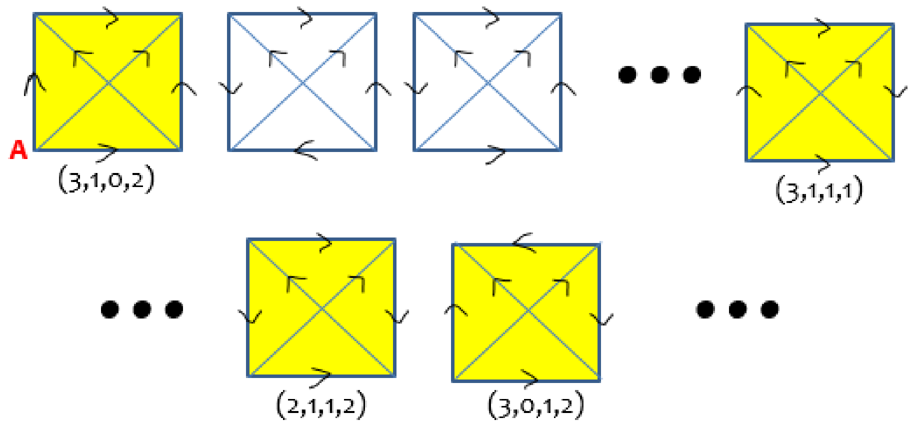


Figure 4. Tournaments on K_3. Highlighted graphs mark score vectors; (2,1,0) and (2,0,1) are treated as identical.

- Among the total $2^{\binom{4}{2}} = 64$ tournaments on K_4, 35 were counted to have vertex A as a transitive champion. As shown in **Fig. 5**, 16 of these 35 score vectors were unique: (3,1,0,2), (3,1,1,1), (2,1,1,2), (3,0,1,2), (3,0,2,1), (2,1,2,1), (2,0,2,2), (1,2,1,2), (1,2,2,1), (1,1,2,2), (3,2,0,1), (2,2,0,2), (3,2,1,0), (2,2,1,1), (3,1,2,0), and (2,2,2,0).

Figure 5. Tournaments on K_4; highlighted graphs mark score vectors. (3,1,0,2) and (3,0,1,2), for example, are treated as identical.



We see from the above examples that the total number of unique score vectors of tournaments on K_n in which a given vertex is a transitive champion seems to be n^{n-2} – a connection to Cayley's famous tree formula! To prove this conjecture, we start by organizing the score vectors according to A's score so as to possibly reveal any hidden patterns:

{1,1,1}	{0,1,2}	{1,1,2}	{0,2,2}	{1,2,2}
(3,1,1,1)	(3,1,0,2)	(2,1,1,2)	(2,0,2,2)	(1,2,1,2)
	(3,0,1,2)	(2,1,2,1)	(2,2,0,2)	(1,2,2,1)
	(3,0,2,1)	(2,2,1,1)	(2,2,2,0)	(1,1,2,2)
	(3,2,0,1)			
	(3,2,1,0)			
	(3,1,2,0)			

	{0,1}	{1,1}
$n=3$	(2,0,1)	(1,1,1)
	(2,1,0)	

Figure 6. Score vectors arranged by A's score. The numbers included within the red brackets above each column are permutated in their corresponding columns, excluding A's score.

Notice that by arranging the score vectors by the convention presented in **Fig. 6**, the following patterns are revealed. (The values to the left of the colon correspond to A's score.)

- For groups in n = 3
  - 2: all permutations of {0,1} sum to 1 = 3 – 2
  - 1: all permutations of {1,1} sum to 2 = 3 – 1

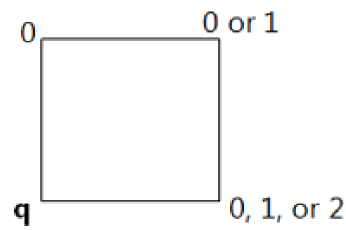
- For groups in n = 4
  - 3a & 3b: all permutations of {1,1,1} and {0,1,2} sum to 3 = 6 – 3
  - 2a & 2b: all permutations of {1,1,2} and {0,2,2} sum to 4 = 6 – 2
  - 1: all permutations of {1,2,2} sum to 5 = 6 – 1

Each case consists of a value for A followed by *all* permutations of some given set of values for B, C, and D. Moreover, we are getting exactly *all* such sequences (i.e. every possibility is being accounted for). If these patterns hold for all n, then counting score vectors of the desired type would be reduced to a concrete combinatorial problem involving sums of factorials; however, this is a point we must check.

Furthermore, investigating the sums of the permutations of the unique score vectors, we find that our problem has now become to use the digits $\{0, 1, 2, \dots, n-2\}$ to count the number of permutations of size $n-1$ that add to $\binom{n}{2} - k$, where A's score is k, and k ranges from 1 to $n-1$ inclusive. This should come to be n^{n-2}; noting the presence of this same expression within Cayley's formula, we ask whether there is a way to associate a unique score vector with A as a transitive winner to each spanning tree in K_n. A 1-1 correspondence of this type would show that the number of such score vectors equals the number of spanning trees, an important finding.

2 ***Counting spanning trees on complete graphs and parking functions.*** We know by Cayley's formula that the total number of spanning trees on K_n is n^{n-2}. However, counting them manually, in the following manner, enables for us to directly associate them with score vectors.

Taking K_4 as an example, we label a regular polygon with n sides as given to the right. Without accounting for rearrangements, we get the five sequences: 0-0-0, 0-0-1, 0-0-2, 0-1-1, and 0-1-2. Once we account for the different possible orders of these sequences (i.e. the number of spanning trees), we get $1 + 3 + 3 + 3 + 6 = 16 = 4^{4-2}$.



Through parking functions, we find a way to associate a unique score vector with A as a transitive winner to each spanning tree in K_n. Taking the graph K_5, we expect to count $5^{5-2} = 125$ spanning trees. Below, we generate all possible 4-term sequences of the form $(s_0, \dots, s_{n-2})$, where each s_i is $\leq i$ for all i – the set of parking functions on K_5. The integers on the right side of the arrows indicate the number of possible rearrangements of the left-hand-side sequence. These integers sum to 125, as expected. However, will a generation of all possible unique score vectors on K_5 yield $n^{n-2} = 5^{5-2} = 125$ as well, as it did for the cases $n = 3$ and 4?

0-0-0-0 → 1	**0-0-2-2** → 6	
0-0-0-1 → 4	**0-0-2-3** → 12	
0-0-0-2 → 4	**0-1-1-1** → 4	
0-0-0-3 → 4	**0-1-1-2** → 12	parking functions
0-0-1-1 → 6	**0-1-1-3** → 12	
0-0-1-2 → 12	**0-1-2-2** → 12	
0-0-1-3 → 12	**0-1-2-3** → 24	

Notice how the number of the sequences (without rearrangements) is **2** for $n = 3$, **5** for $n = 4$, and **14** for $n = 5$. These are the Catalan numbers! In our case, the Catalan number that corresponds to the sequence count (without rearrangements) for a given n is $\frac{1}{n}\binom{2n-2}{n-1}$. The question now becomes how to count the total number of *rearrangements* of the $\frac{1}{n}\binom{2n-2}{n-1}$ parking functions (i.e. the number of spanning trees).

In an attempt to establish a 1-1 correspondence between unique score vectors of tournaments with transitive champions and rearrangements of parking functions, we list the following for the 125 spanning trees on K_5 as a test to see whether we achieve, like for K_3 and K_4, matching counts. Since the total possible number of directed tournaments on K_5 ($2^{\binom{5}{2}} = 1024$) is too large to depict manually, we provide a simple listing of all *potential* score vectors. (For each of K_3 and K_4, the set of all potential score vectors was identical to the set of actual score vectors.) The results reveal an important problem (see **Fig. 7** description), which we seek to amend by introducing the concept of *break divisors*, an amazing mathematical tool for solving graph theory problems.

Score Vector	Rearrangements
(4) 0-0-3-3	6
(4) 0-1-2-3	24
(4) 0-2-2-2	4
(4) 1-1-1-3	4
(4) 1-1-2-2	6
(3) 0-1-3-3	12
(3) 0-2-2-3	12
(3) 1-1-2-3	12
(3) 1-2-2-2	4
(2) 0-2-3-3	12
(2) 1-1-3-3	6
(2) 1-2-2-3	12
(2) 2-2-2-2	1
(1) 0-3-3-3	4
(1) 1-2-3-3	12
(1) 2-2-2-3	4

Figure 7. Score vectors, grouped by vertex A's score, and corresponding number of sequence rearrangements. Note that instead of 14 unique score vectors and 125 total rearrangements (i.e. spanning trees), the table to the left shows that there are 16 different score vectors and 135 total rearrangements.

We conclude that there must be some score vectors that could not possibly represent tournaments on complete graphs; specifically in this case, 2 score vectors and 10 rearrangements are extraneous. By translating the score vectors into the language of break divisors, we find that the following score vectors and their corresponding break divisor forms do not actually qualify as break divisors: (4) 0-0-3-3 → (0,0,0,3,3) and (1) 0-3-3-3 → (0,0,0,2,4), due to a violation of the characteristics of break divisors.[7] Applying a similar conversion to break divisors upon the score vectors on K_6, we find that the sequences marked in red could not be possible score vectors of tournaments in which A is a transitive champion. As shown in **Fig. 8**, we find that there are 49 sequences, though ideally there should be 42 (or 1296 rearrangements) that work.

A **= 5**	***A*** **= 4**	***A*** **= 3**	***A*** **= 2**	***A*** **= 1**
(5) 0-0-2-4-4	(4) 0-0-3-4-4	(3) 0-0-4-4-4	(2) 0-1-4-4-4	(1) 0-2-4-4-4
(5) 0-0-3-3-4	(4) 0-1-2-4-4	(3) 0-1-3-4-4	(2) 0-2-3-4-4	(1) 0-3-3-4-4
(5) 0-1-1-4-4	(4) 0-1-3-3-4	(3) 0-2-2-4-4	(2) 0-3-3-3-4	(1) 1-1-4-4-4
(5) 0-1-2-3-4	(4) 0-2-2-3-4	(3) 0-2-3-3-4	(2) 1-1-3-4-4	(1) 1-2-3-4-4
(5) 0-1-3-3-3	(4) 0-2-3-3-3	(3) 0-3-3-3-3	(2) 1-2-2-4-4	(1) 1-3-3-3-4
(5) 0-2-2-2-4	(4) 1-1-1-4-4	(3) 1-1-2-4-4	(2) 1-2-3-3-4	(1) 2-2-2-4-4
(5) 0-2-2-3-3	(4) 1-1-2-3-4	(3) 1-1-3-3-4	(2) 1-3-3-3-3	(1) 2-2-3-3-4
(5) 1-1-1-3-4	(4) 1-1-3-3-3	(3) 1-2-2-3-4	(2) 2-2-2-3-4	(1) 2-3-3-3-3
(5) 1-1-2-2-4	(4) 1-2-2-2-4	(3) 1-2-3-3-3	(2) 2-2-3-3-3	
(5) 1-1-2-3-3	(4) 1-2-2-3-3			
(5) 1-2-2-2-3	(4) 2-2-2-2-3			
(5) 2-2-2-2-2				

Figure 8. Score vectors (without rearrangements) for K_6.

Thus, we see that when mapping unique score vectors to spanning trees on K_n, a generation of all possible parking function rearrangements is indeed n^{n-2}, but a similar generation of all possible unique score vectors is slightly larger than n^{n-2} for $n > 4$:

$$\boxed{n^{n-2}} = \boxed{\text{Number of rearrangements of parking functions}} = \boxed{\text{Number of unique score vectors}} - \boxed{\text{Some quantity dependent upon } n}$$

We may represent the number of unique score vectors (without the subtraction correction), which can be defined by the property (i) of break divisors, by a generating function. Suppose we divide $\binom{n-1}{2}$ objects into n boxes such that each box can have up to $n-2$ objects. It can be shown that the number of divisors[8] (not necessarily *break* divisors, as property (ii) has not been accounted for) on a complete graph with entries between 0 and $n-2$ inclusive that sum to $\binom{n-1}{2}$ is equivalent to the coefficient of $x^{\binom{n-1}{2}}$ in the generating function

$$(1 + x + x^2 + \cdots + x^{n-2})^n = \left(\frac{1 - x^{n-1}}{1 - x}\right)^n,$$

where $\frac{1}{(1-x)^n} = \binom{n-1}{n-1} + \binom{n}{n-1}x + \binom{n+1}{n-1}x^2 + \cdots$.

When expanded out, we get 16 for $n = 4$ and 135 for $n = 5$ as expected. We look to the properties of break divisors – rather, we set them to be class representatives of equivalence classes defined by the Cori-LeBorgne result – to find a way to eliminate sequences that do not qualify as break divisors, through the incorporation of property (ii).

3 ***Fundamental cycles and an explicit inverse map.*** In the "Appendix" of the report, we attempt to establish a bijection between spanning trees and break divisors. It was found that we may achieve this by tackling both directions (spanning trees → break divisors and break divisors → spanning trees) in two different ways.

4 ***Equivalence classes and break divisors as class representatives.*** Here, we follow the map provided in the "Research Objective" section.

We begin by showing that there are exactly n^{n-2} equivalence classes of divisors of any given degree. The definition of equivalence we use here is that two sequences $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ of integers (i.e. two "divisors") are equivalent if and only if $\sum a_i = \sum b_i$ and $a_i - a_j \equiv b_i - b_j \pmod{n}$ for all i, j. The goal is to show that, restricting to divisors with sum of entries equal to $\binom{n-1}{2}$, there is exactly one break divisor in each equivalence class, and the number of equivalence classes is n^{n-2}.

We begin by noting that every n-tuple of integers $(a_1, \ldots, a_n)$ with a fixed sum is equivalent to one in which $a_1 = 0$, and then furthermore to one in which $a_i \in \{0,1, \ldots, n-1\}$ for all $i = 2, \ldots, n-1$. Thus, the configuration becomes $(0, a_2, \ldots, a_n)$, where the value of a_n is determined by the values of $a_2 \dots a_{n-1}$, since the sum of the $\sum a_i$ is a fixed value g. Moreover, it is easy to check that two such n-tuples $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are equivalent if and only if they are equal: substituting $j = 1$ into the equivalence class definition, we get that $a_i \equiv b_i \pmod{n}$. Since the values of a_i and b_i are restricted to the range $[0, n-1]$, a_i must be equal to b_i for the congruence to be held. Thus, $a_i = b_i$ for all i. For i ranging from 2 to $n-1$ ($n-2$ choices), there are n^{n-2} ways to construct pairs of chip-firing equivalent divisors, so there are n^{n-2} equivalence classes of any given degree. ☐

Suppose that $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are two sequences of integers satisfying the break divisor properties (i) and (ii) such that $a \sim b$, i.e. a and b are equivalent with respect to the established equivalence relation. We claim that $a = b$. To see this, we can assume without loss of generality that both a and b are sorted in increasing order. Then $0 \leq a_k - a_1 \leq n - 2$ for all $k = 1, \dots, n$ and similarly for $b_k - b_1$. Since $a \sim b$, we have by the established equivalence relation that $a_k - a_1 \equiv b_k - b_1 \pmod{n}$ for all k, and thus that $a_k - a_1 = b_k - b_1$ for all k. Summing from $k = 1$ to n gives $(a_1 + \cdots + a_n) - na_1 = (b_1 + \cdots + b_n) - nb_1$, and because $\sum a_i = \sum b_i = \binom{n-1}{2}$, $a_1 = b_1$. Since $a_k - a_1 = b_k - b_1$ for all k, it now follows that $a = b$ as claimed – this proves that each equivalence class contains *at most one* sequence (i.e. break divisor) satisfying (i) and (ii). ☐

Below, we provide an argument to show that each equivalence class with total sum equal to $\binom{n-1}{2}$ contains *at least one* sequence satisfying property (ii).

- Let $a = (a_1, \dots, a_n)$ be any member of the equivalence class. By rearranging, we may assume without loss of generality that $a_1 \leq \cdots \leq a_n$. Suppose that $a_1 < 0$. Then we must have $a_{k+1} \geq k$ for some $k = 1, \dots, n-1$; otherwise we get a contradiction.

Assuming, for example, that $a_{k+1} < k$ for all k, the greatest possible integer value for a_{k+1} is $k-1$. In turn the largest possible $\sum a_i$ value is

$$(-1+0+1+\cdots+n-2) = \frac{(n-2)(n-1)}{2} - 1 = \binom{n-1}{2} - 1.$$

Since the required $\sum a_i$ value was $\binom{n-1}{2}$, we can see that at the most extreme case, the condition $a_{k+1} < k$ causes $\sum a_i$ to fall too short. Thus, $a_{k+1} \geq k$ for some k in the range $1, \dots, n-1$. $\square$

- We note that we can subtract k from each of $a_{k+1}, \dots, a_n$ and add $n-k$ to each of $(a_1, \dots, a_k)$ without changing the current equivalence class, and the new (re-sorted if necessary) sequence $b' = (b_1, \dots, b_n)$ satisfies $b_1 > a_1$. (This can be rephrased as a chip-firing argument, as we are simultaneously firing vertices $v_{k+1}, \dots, v_n$ in K_n.)
- We continue sorting in this way until we get to an ordered sequence $(c_1, \dots, c_n)$ with all entries non-negative. Next, we use induction on m (starting with the base case $m = 1$ just established) and a similar argument as above to show that for each $m = 1, \dots, n-1$ there is a sorted sequence in the same equivalence class with $s_k \geq \binom{k-1}{2}$ for all $k = 1, 2, \dots, m$.

For the base case $m = 1$, it is calculated that $s_1 \geq \binom{1-1}{2} = 0$ from the sorting rules above. Now we suppose that for $m = k$, $s_1 \geq \binom{1-1}{2} = 0, \dots, s_k \geq \binom{k-1}{2}$ holds true. For $m = k+1$, we will show that $s_{k+1} \geq \binom{k}{2}$. Another way to write this expression is

$$\min(s_{k+1}) = \binom{k-1}{2} + c_{k+1},$$

and the objective becomes to show that $c_{k+1} \geq \binom{k}{2} - \binom{k-1}{2} = k-1$ for all k. Right away, we see that the "all k" condition may not hold true for some sequences (despite being non-negative) in the equivalence class, so we must show that it holds after possibly replacing $(c_1, \dots, c_n)$ with an equivalent sequence.

For example, when dealing with cases in which $c_1 \geq 0$, we take a similar approach as before in deliberately giving a contradiction; supposing that $c_{k+1} < k-1$, the greatest possible integer value for c_{k+1} is $k-2$. Therefore the largest possible $\sum c_i$ value is

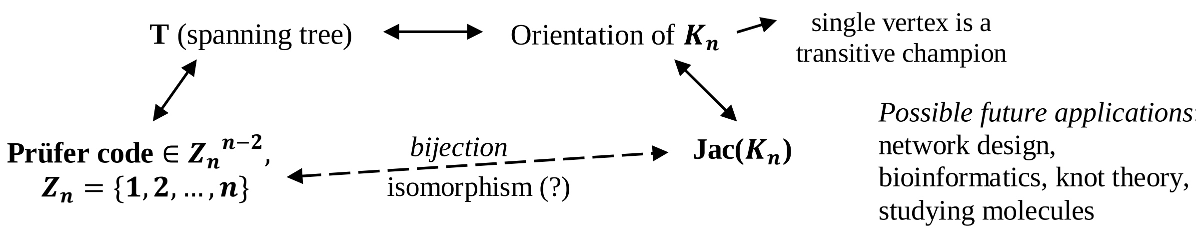
$$(-2+-1+0+\cdots+n-3) = \frac{(n-3)(n-2)}{2} - 3 = \binom{n-2}{2} - 3.$$

Since the required $\sum c_i$ value is $\binom{n-1}{2}$ by definition, we can see that at the most extreme case, the condition $c_{k+1} < k$ causes $\sum c_i$ to be much too small. Hence, we get that $c_{k+1} \geq k-1$ and $\min(s_{k+1}) = \binom{k-1}{2} + k - 1 = \binom{k}{2}$ for some k in the range $1, \dots, n$.

In order to ensure that $c_{k+1} \geq k-1$ for *all* k, we must apply a transformation of sorts. We note that adding $(n-k-1)$ to each of $(c_1, \dots, c_k)$ and subtracting $\frac{(n-k-1)(k)}{n-k}$ from each of $c_{k+1}, \dots, c_n$ for any desired k value, where $1 \leq k \leq n$ and $\frac{(n-k-1)(k)}{n-k}$ is an integer, does not change the equivalence class in which a divisor resides. Repeated transformations and re-sortings yield an equivalent sequence c' for which $c'_{k+1} \geq k-1$ for all k and thus that each equivalence class contains exactly one break divisor. (A concrete example of this concept would be the sequence (0, 1, 1, 1), which is a break divisor but does not fulfill $c_{3+1} \geq 3-1$, for $1 \ngeq 2$. Applying the above transformation and choosing $k=2$, we have (0, 1, 1, 1) → $c' =$ (1, 2, 0, 0) = (0, 0, 1, 2), which does fulfill $c'_{k+1} \geq k-1$ for all k.) $\square$

Conclusions, Applications to IoT, and Future Work

The results obtained from this research project have allowed us to characterize break divisors in a unique and compact form by properties (i) and (ii). By relating them to spanning trees on K_n, we are able to reduce the complex counting problem to a slightly simpler problem. We also confirm that the relationship between spanning trees and break divisor n-tuples is indeed a bijection. Combined with the proof that each equivalence class of divisors contains exactly one break divisor, we have given a new proof of Cayley's theorem. Furthermore, we show that the number of score vectors of tournaments in which a given player A is a transitive champion is n^{n-2}. Future work includes exploring the relationship between the Jacobian of K_n and Prüfer codes[10,11] and extending the concept of break divisors represented as n-tuples to non-complete graphs. A map of the general idea is given here:



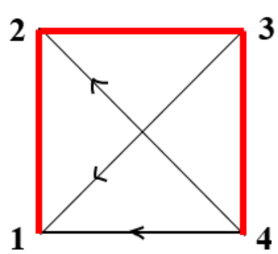
Distinctive ideas and concepts explored in this report include break divisors on complete graphs, transitive champions and score vectors, and partial orientations to establish a bijection between such break divisors and spanning trees. Thus, this research has the potential to aid in the construction of reliable multi-state flow networks in a multitude of ways. Examples include infrastructure applications within the Internet of Things (IoT), in which one system is dominant over the others. Such designs and structures can be modeled by score vectors for tournaments with transitive champions. Healthcare applications include modeling the possible stages of cancer movement through partial graph orientations to ensure early detection. Regarding break divisors as identification tools for spanning trees may help support the interaction between "things", allowing for the development of complex structures through distributed computing. Furthermore, the IoT, a sensor-based system, is expected to generate large amounts of data from diverse locations that is aggregated very quickly, thereby increasing the need to better index, store, and process such data. Defining break divisors as number sequences, efficient algorithms can be created to accomplish such feats.

Appendix

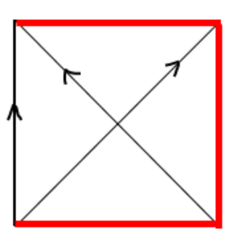
Below we propose two conjectures, which we hope to soon further develop into geometric direct bijections between integer n-tuples satisfying (i) and (ii), i.e. break divisors, and spanning trees on complete graphs, without making use of intermediaries such as parking functions or Prüfer codes.

Fundamental cycles and an explicit inverse map. Here we attempt to establish a bijection between spanning trees and break divisors. It was found that we may achieve this by tackling both directions (spanning trees → break divisors and break divisors → spanning trees) in two different ways.

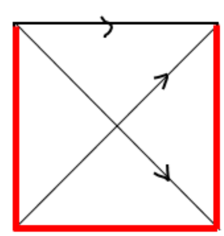
Fundamental cycles are defined to be cycles formed after adding a single edge to a spanning tree. Because there is a distinct fundamental cycle for each edge, there is a 1-1 correspondence between fundamental cycles and edges not in the spanning tree. Below we employ a method of associating break divisors to spanning trees that takes advantage of mixed graphs, or partially directed graphs, and captures the "essence" of fundamental cycles. The undirected edges in red on each graph below represent the spanning tree, while the edges in black (i.e. those not in the spanning tree) are directed such that each black edge, capped by two red endpoints from which two or more red segments emanate and forming a cycle with said red segments, will be oriented in a direction away from the smallest vertex in that cycle. Below we show this assignment on the spanning trees of K_4. The break divisors included below each graph consist of all possible sequences of four integers within the range [0,2] that sum to 3.



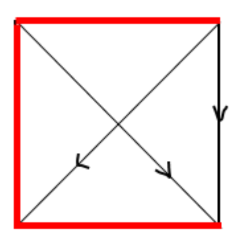
(2,1,0,0)



(0,2,1,0)



(0,0,2,1)



(1,0,0,2)

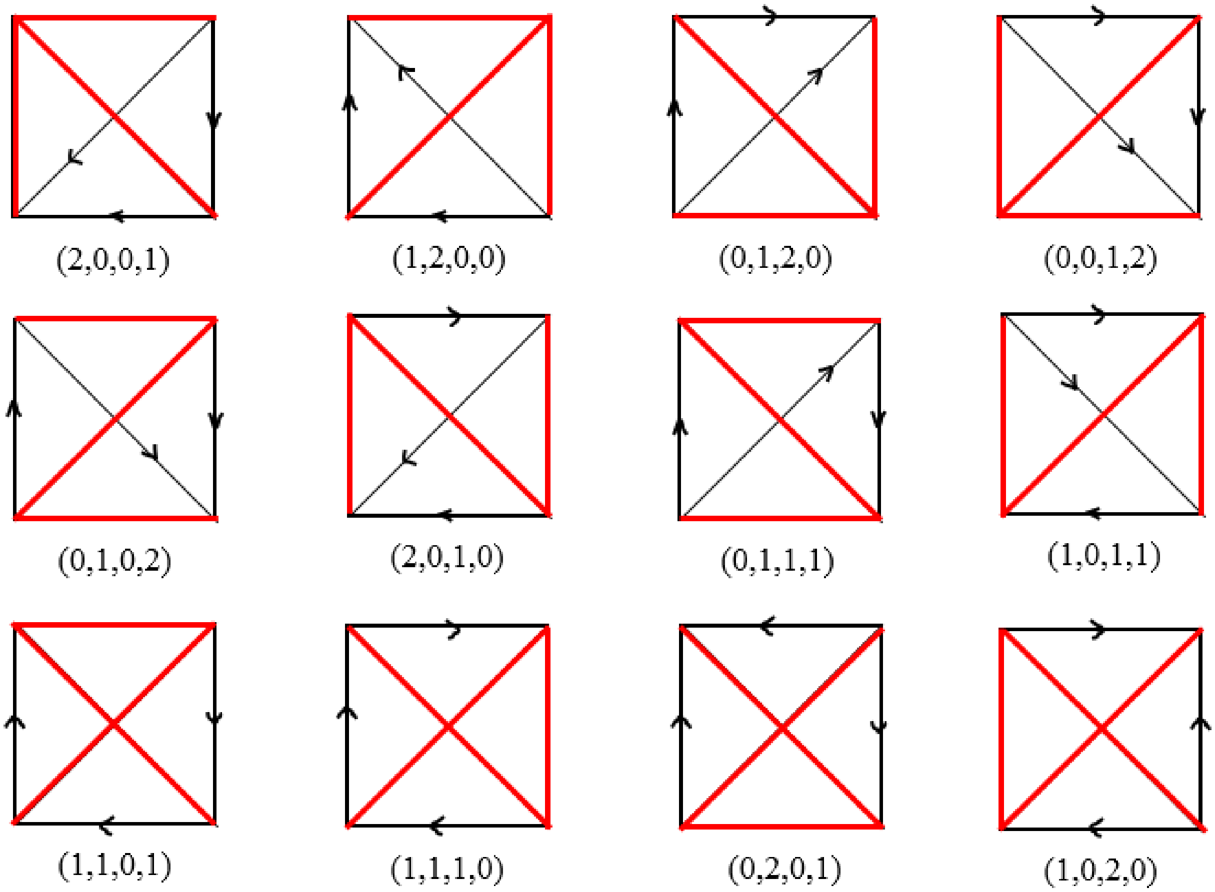


Figure 9. Associating break divisors to spanning trees on K_4 via mixed graphs.

Now we tackle the reverse direction. Below we give an algorithm serving as an explicit inverse to the break divisors $\rightarrow$ spanning trees bijection. Labeling the vertices of K_n as $v_1, \dots, v_n$, we have the following. We conjecture that the algorithm gives an inverse mapping for all n.

```
Input: A break divisor D.
Initialize B := D, S := v_n, T := empty set, and H := G.

WHILE S does not contain all of {v_1,...,v_n} DO
     Let v be the LARGEST vertex not in S which is
     connected to S by an edge in H.
     Let e = vw be the SMALLEST edge in H connecting v to
     some w in S.

     IF deleting e from H and subtracting 1 chip from B
     at v' gives a break divisor B' on H' = H − e
          Set H := H' and B := B'.
     ELSE
          Set S := S ∪ v and T := T ∪ e.

Output: the spanning tree T.
```

Figure 10. Algorithm for geometric bijection.

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